Income Distribution and Macroeconomics

The Galor-Zeira Model

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- Production in the unskilled-intensive sector:

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Factor Prices

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$\Longrightarrow$

$$(r_t, w_t^s, w_t^u) = (r, w^s, w^u) \qquad \forall t$$

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- Differ in:

  Parental income $\Rightarrow$ Inv't in HC

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- Second period of life (Period $t + 1$):
  - [work as unskilled / no inv't in HC] or [work as skilled / inv't in HC]

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Second period budget constraint:

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$\omega_{t+1} \equiv$ wealth in period $t+1$

Member of Generation t: Optimization

$$\{c_{t+1}, b_{t+1}\} = \arg\max[\alpha \ln c_{t+1} + (1-\alpha) \ln b_{t+1}]$$

$$\text{s.t.} \qquad c_{t+1} + b_{t+1} \leq \omega_{t+1}$$

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$\Longrightarrow$ v^t is monotonic increasing in 2nd period wealth, ω_{t+1}

$\Longrightarrow$ maximization of ω_{t+1}, is the basis of occupational choices

Fundamental Assumptions

- Imperfect Capital Markets:

$$r < i \tag{A1}$$

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- Fixed cost of education (Indivisibility of inv't in HC)

$$h > 0 \tag{A2}$$

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$$\omega_{t+1}^{u} \quad =$$

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Assumptions

- Investment in human capital is *not* beneficial for individuals who must finance the entire cost of education via borrowing

$$w^s - (1+i)h < 0 \tag{A3}$$

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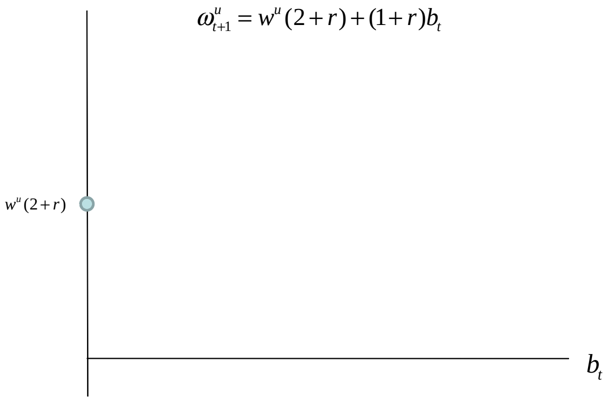
- Investment in human capital is beneficial for individuals who can finance the entire cost of education *without* borrowing

$$w^s - (1+r)h > w^u(2+r) \tag{A4}$$

Income from Being Unskilled Worker



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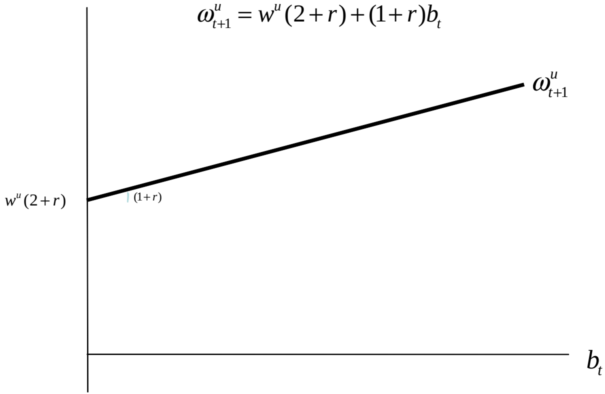
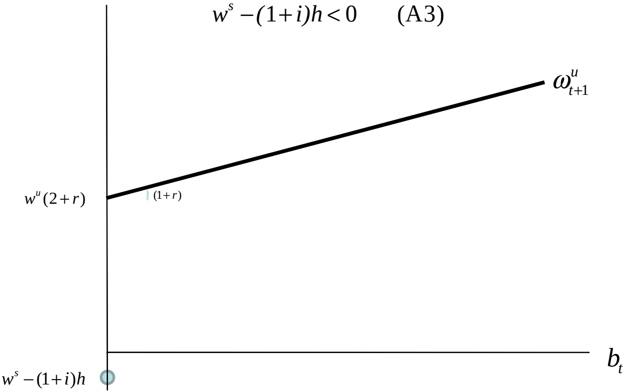


Income from Being Skilled Worker: Borrowers



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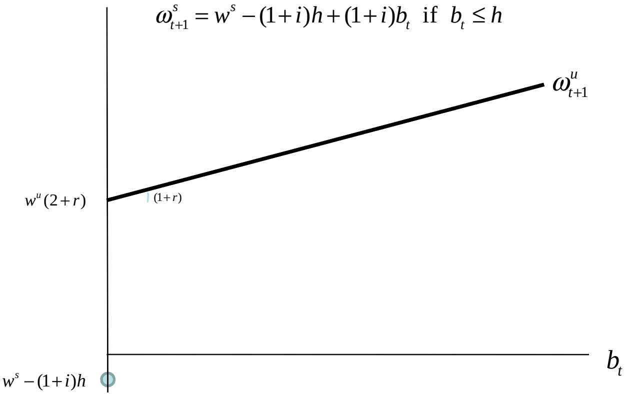

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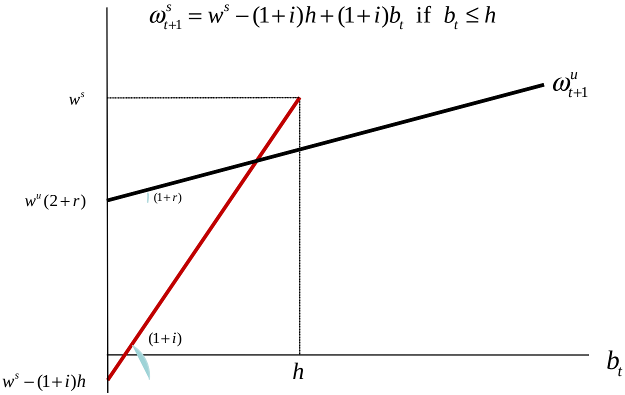

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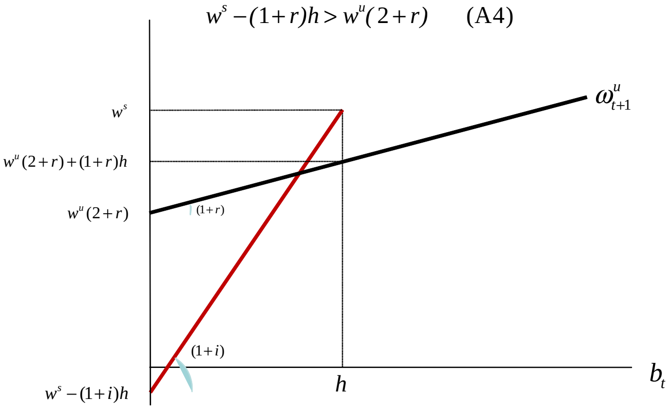
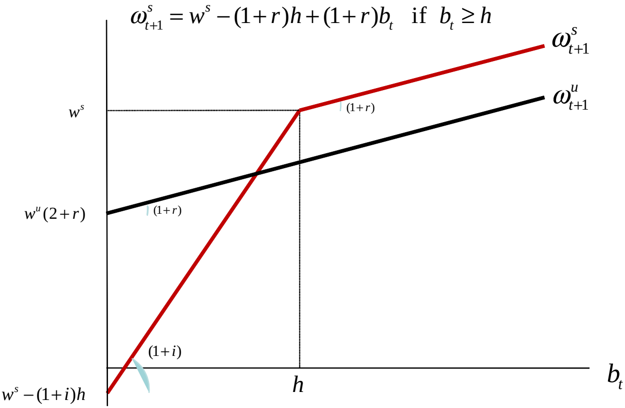
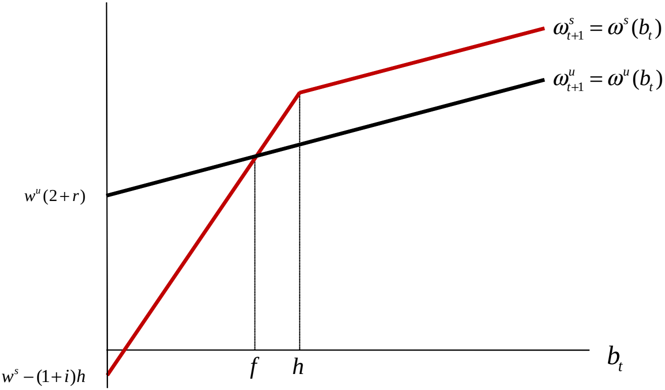


Income from Being Skilled Worker: Lenders



Bequest and Occupational Choice



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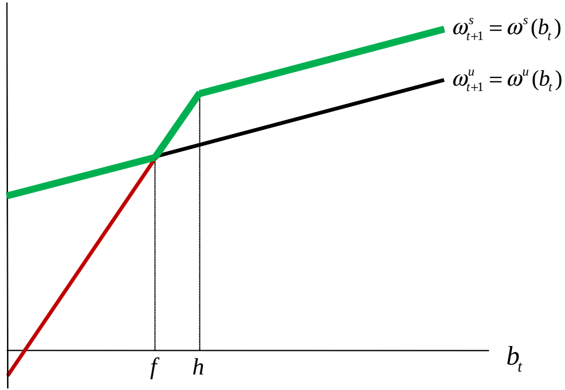

Bequest and Occupational Choice

$$b_t \quad \begin{cases} < f & \rightarrow \omega^u_{t+1} > \omega^s_{t+1} \text{ (individual } t \text{ becomes unskilled)} \\ > f & \rightarrow \omega^u_{t+1} < \omega^s_{t+1} \text{ (individual } t \text{ becomes skilled)} \end{cases}$$

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where

$$f = \frac{w^u(2+r) - [w^s - (1+i)h]}{i - r} > 0$$

Bequest Dynamics

b_{t+1}

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Bequest Dynamics: Sufficient Conditions for Multiplicity of Steady-Sate

$$
\begin{aligned}
(1-\alpha)(1+r) &< 1 \\
(1-\alpha)(1+i) &> 1
\end{aligned}
\tag{A5}
$$

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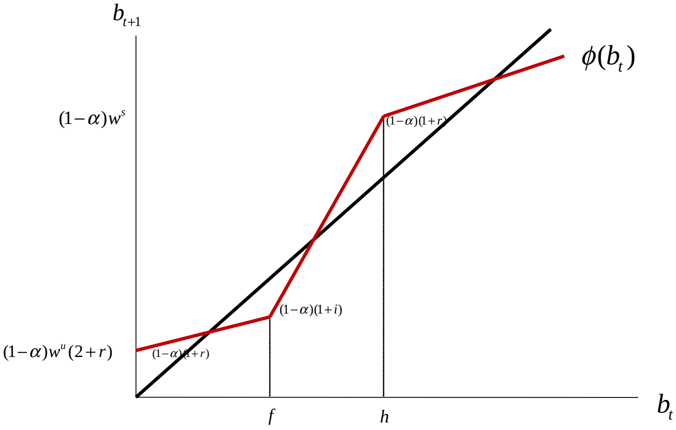
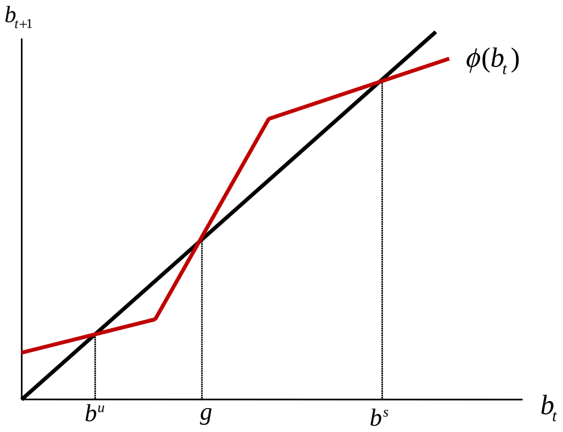
$$(1-\alpha)w^s > h \tag{A6}$$

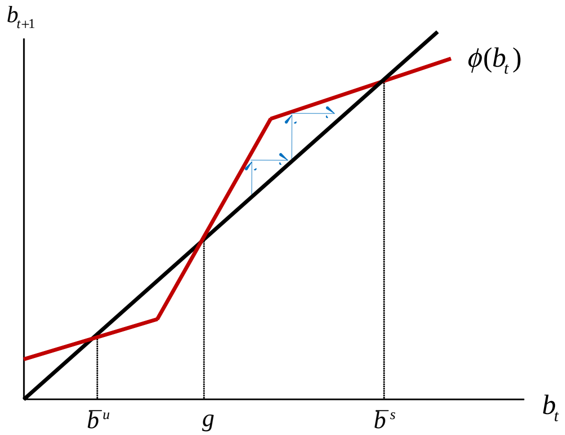
Bequest Dynamics



Bequest Dynamics: Multiple Steady-State Equilibrium



Bequest Dynamics: Stability of High Bequest Equilibrium



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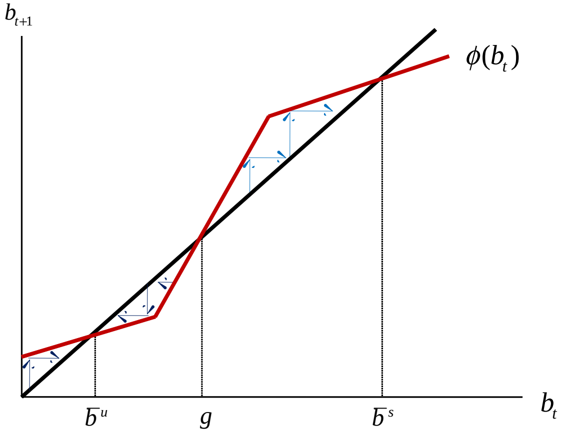
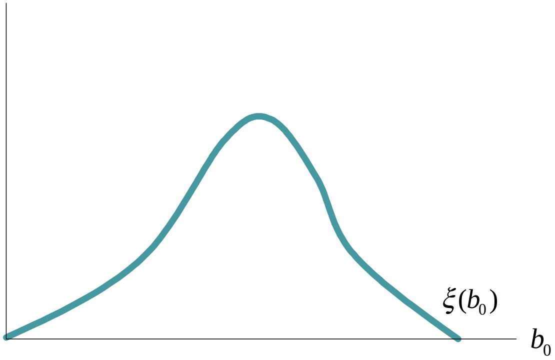


The Distribution of the Inheritance in Period t



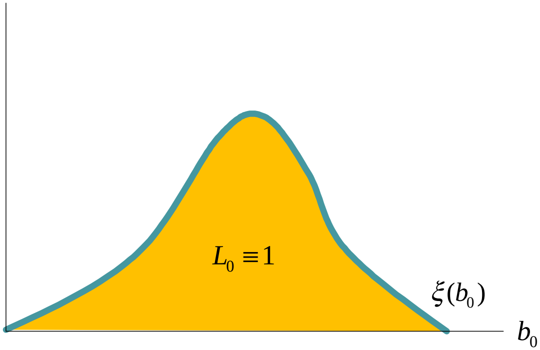
Income Distribution and the Long Run Decomposition of the Labor Force

$\xi_t(b_t) \equiv$ Distribution of inheritance at time t

$\Longrightarrow$

$$L_t = \int_0^\infty \xi(b_t) db_t \equiv 1$$

The Distribution of the Inheritance in Period t



Income Distribution of the Long Run Decomposition of the Labor Force

$$\lim_{t\to\infty} l_t^u = \int_0^g \xi_t(b_t)db_t \equiv \bar{l}^u$$

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where

$$\partial\bar{l}^s/\partial g < 0$$

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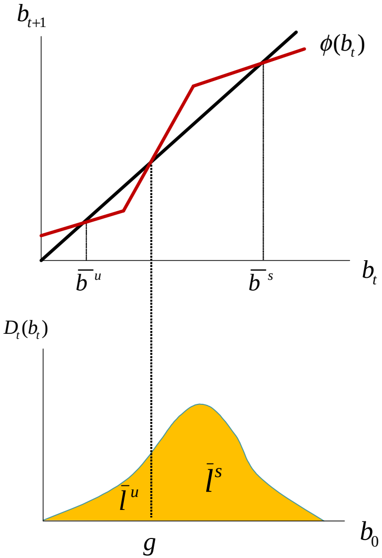
where

$$\partial\bar{l}^s/\partial g < 0$$

and

$$g = \frac{(1-\alpha)[(1+i)h - w^s]}{(1-\alpha)(1+i)-1} > 0$$

Income Distribution of Skill Composition



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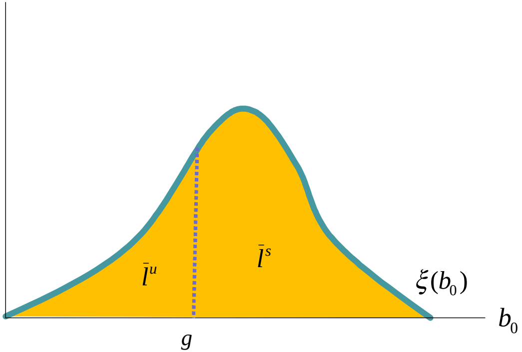
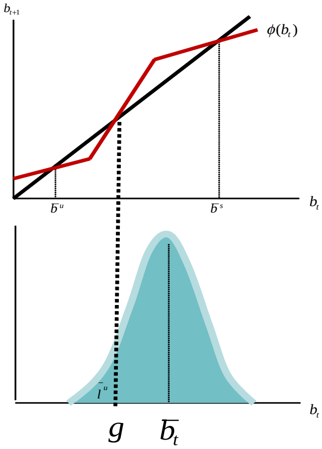
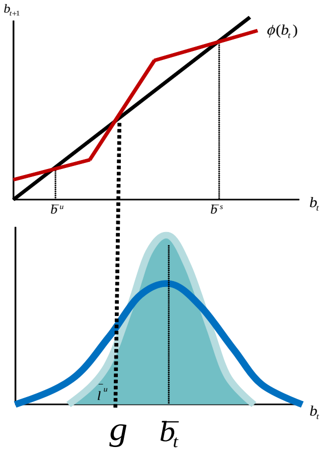


Inequality and Development: Rich Economies



Rich Economies: Inequality is Harmful for Development

Inequality reduces human capital formation



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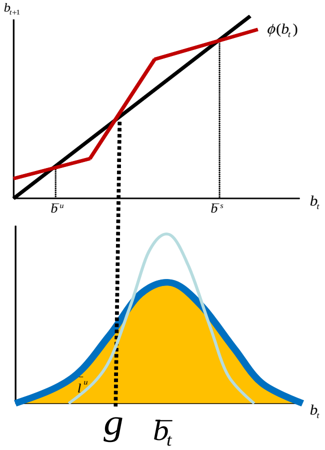
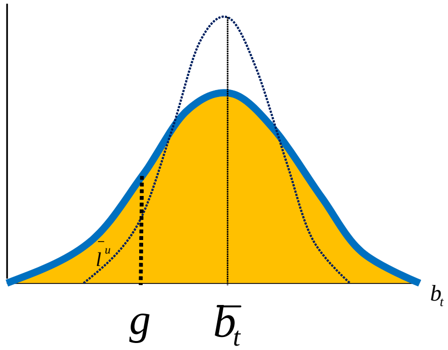
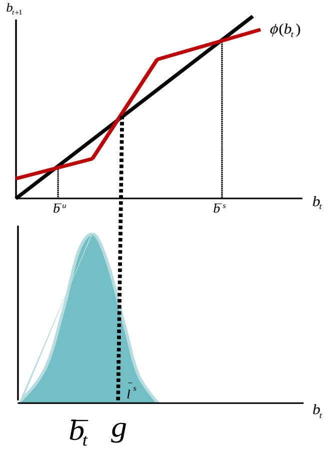


Rich Economies: Inequality is Harmful for Development

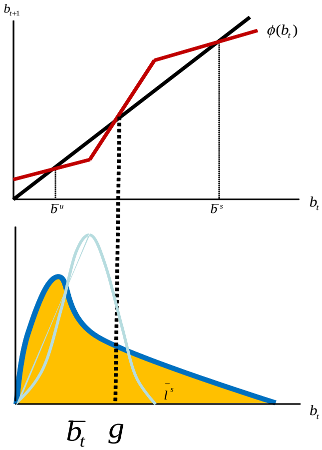


Inequality and Development: Poor Economies

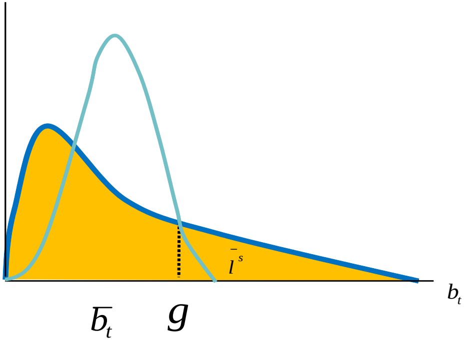


Poor Economies: Inequality may Benefit Development

Inequality stimulates human capital formation



Poor Economies: Inequality may Benefit Development



Robustness

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- Concave production function of human capital

Robustness: Technological Progress and Endogenous Education Cost

Labor Augmenting Technological Progress: increases the productivity of workers in both the skilled-intensive and the unskilled intensive sector.

- Production in the skilled-intensive sector

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- Technological progress

$$A_{t+1} = (1+\lambda) A_t \qquad \lambda > 0.$$

Robustness: Technological Progress and Endogenous Education Cost

Factor Prices

$$w_t^s = A_t[f(k) - f'(k)k] \equiv A_t w^s$$

$$w_t^u = A_t a \equiv A_t w^u$$

$$r_t = r$$

Cost of Education

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$$C_t^H = \theta A_t w^s + (1-\theta) A_t w^u \equiv A_t h$$

Income: Unskilled Individuals

$$x^u_{t+1} = (A_t w^u + b_t)(1 + r) + A_{t+1} w^u$$

Income: Unskilled Individuals

$$
\begin{aligned}
x_{t+1}^{u} &= (A_t w^u + b_t)(1+r) + A_{t+1} w^u \\
&= A_t w^u (2 + r + \lambda) + (1+r) b_t
\end{aligned}
$$

Income: Skilled Individuals

$$x^s_{t+1} = \begin{cases} A_{t+1}w^s - (A_t h - b_t)(1+i) & \text{if} \quad b_t \leq A_t h \\ A_{t+1}w^s + (b_t - A_t h)(1+r) & \text{if} \quad b_t \geq A_t h \end{cases}$$

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$$\Longrightarrow$$

$$x_{t+1}^{s} = \begin{cases} A_t[w^{s}(1+\lambda) - (1+i)h] + (1+i)b_t & \text{if} \quad b_t \leq A_t h \\ A_t[w^{s}(1+\lambda) - (1+r)h] + (1+r)b_t & \text{if} \quad b_t \geq A_t h \end{cases}$$

Threshold level of Bequest for Becoming Skilled Worker in Period t

$$f = \frac{A_t\{w^u(2+r) - [w^s - (1+i)h] - \lambda(w^s - w^u)\}}{(i-r)}$$

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for

$$w^u(2+r) > [w^s - (1+i)h] + \lambda(w^s - w^u)$$

Bequest Dynamics

$$b_{t+1} = \begin{cases} (1-\alpha)\{A_t w^u (2+r+\lambda) + (1+r) b_t\} & b_t \in [0, f] \\ (1-\alpha)\{A_t [w^s (1+\lambda) - (1+i) h] + (1+i) b_t\} & b_t \in [f, A_t h] \\ (1-\alpha)\{A_t [w^s (1+\lambda) - (1+r) h] + (1+r) b_t\} & b_t \in [A_t h, \infty] \end{cases}$$

Bequest Dynamics

Let $\hat{b}_{t+1} \equiv b_{t+1}A_{t+1}$

$$\hat{b}_{t+1} = \begin{cases} \left[\frac{1-\alpha}{1+\lambda}\right]\left\{w^u(2+r+\lambda)+(1+r)\hat{b}_t\right\} & \hat{b}_t \in [0,(\hat{f})] \\ \left[\frac{1-\alpha}{1+\lambda}\right]\left\{[w^s(1+\lambda)-(1+i)h]+(1+i)\hat{b}_t\right\} & \hat{b}_t \in [\hat{f},h] \\ \left[\frac{1-\alpha}{1+\lambda}\right]\left\{[w^s(1+\lambda)-(1+r)h]+(1+r)\hat{b}_t\right\} & \hat{b}_t \in [h,\infty] \end{cases}$$

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$\Rightarrow$ The dynamical system is unaffected qualitatively by labor-augmenting technological progress

Sufficient Conditions for Multiple Steady-States

$$(1-\alpha)(1+r) < (1+\lambda)$$

$$(1-\alpha)(1+i) > (1+\lambda)$$

$$w^s(1+\lambda) - (1+i)h < 0$$

$\Rightarrow$ The system is characterized by multiple steady-state, where the unstable equilibrium

$$\hat{g} = \frac{(1-\alpha)[(1+i)h - w^s(1+\lambda)]}{[(1-\alpha)(i+i) - (1+\lambda)]} > 0$$

Income Per Capita in the Long Run

- Income of a skilled individual in the second period of life (wage and capital income)

$$I_2^s = w^s + (\bar{b}^s - h)r$$

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- Income of an unskilled individual in the first period of life (only wage income)

$$I_1^u = w^u$$

Income Per Capita in the Long Run

- Aggregate income in the steady-state

$$\bar{Y} = l_2^s \bar{l}^s + l_2^u \bar{l}^u + l_1^u \bar{l}^u$$

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$$\begin{aligned} Y &= [w^s - rh + r\bar{b}^s]\bar{l}^s + [w^u(2+r) + r\bar{b}^u](1 - \bar{l}^s) \\ &= w^u(2+r) + r\bar{b}^u + [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]\bar{l}^s \end{aligned}$$

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- Income per capita

$$\bar{y} = \bar{Y}/2$$

Skill Composition and Income Per Capita in the Long Run

- An increase in the fraction of skilled workers increases income per capita in the steady-state

$$\frac{\partial \bar{y}}{\partial \bar{l}^s} = [(w^s - rh) - w^u(2+r) + (\bar{b}^s - \bar{b}^u)]/2 > 0$$

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$$\bar{b}^s > \bar{b}^u$$

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since

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$$\bar{b}^s > \bar{b}^u$$

- An increase in g reduces income per capita in the steady-state

$$\frac{\partial \bar{y}}{\partial g} = \frac{\partial \bar{y}}{\partial \bar{l}^s}\frac{\partial \bar{l}^s}{\partial g} < 0$$